(Def^n.) <u>Compact Space</u> - Let X be a metric space. A Collection $\{G_\alpha\}$ of open subsets of X is called an open cover of X if $X = \cup G_\alpha$.

A metric space X is said to be <u>Compact</u> if every open cover has a finite subcover.

~~Model~~ ~~KA~~ MSc M.U. 9T

QNo → Show that a Compact set in a metric space is closed.

or, QNo → A Compact subset of a metric space is closed.

<u>Proof</u>:- Let K be a compact subset of a metric space X. Let $p \in K^c$. Then for each $q \in K$, $d(p,q) > 0$. We consider n'hoods $V_q$ and $W_q$ of p and q respectively with centeres at p and q and radius less than $\frac{1}{2} d(p,q)$. If we allow q to vary over K, we get a collection $\{W_q\}$ of open subsets which covers K. Since K is compact, this open cover has a finite subcover. In other words there exist finite no. of points $q_1, q_2, \text{---}, q_n$ such that,

$$K \subseteq W_{q_1} \cup \text{----} \cup W_{q_n}.$$

If we put $V = V_{q_1} \cap \text{----} \cap V_{q_n}$ then V is a n'hood of p which doesnot intersect K, i.e. $V \subset K^c$. Hence $K^c$ is open. This proves that K is closed.

Q.No→ Define a metric space and give its at least two examples.

Ans. (Defn.) Metric space :- A metric space is a non-empty set X together with a metric function $d : X \times X \rightarrow \mathbb{R}$. Satisfying the following Condition :-

(i) $d(p,q) \geq 0$ and $d(p,q) = 0$ iff $p = q$.

(ii) $d(p,q) = d(q,p)$

(iii) $d(p,q) \leq d(p,r) + d(r,q)$ for all $p, q, r \in X$.

Example -(i) The real number System $\mathbb{R}$ is a metric space under the metric defined as,

$$d(x,y) = |x - y|$$

Example (ii) The space $\mathbb{R}^n$ of n - tuples of real numbers is a metric space under the metric defined as,

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

where, $x = (x_1, ----, x_n), y = (y_1, ----- y_n) \in \mathbb{R}^n$.

$$d(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

(Defn) N'hood (Neighbourhood) :- Let $X$ be a metric space with metric $d$. Let $p \in X$.

Then the set,

$$S_r(p) = N_r(p) = \{q \in X : d(q,p) < r\}$$

is called an open n'hood of $p$ with centre at $p$ and radius $r$.

(Defn) Interior Point :- Let $E \subset X$ and $p \in E$. Then $p$ is called an interior point of $E$ if there is a real number $r > 0$ such that $N_r(p) \subset E$.

The set of all interior points of $E$ is called the interior of $E$ and is denoted by $E^0$.

A subset $E$ of $X$ is open if every point of $E$ is an interior point of $E$.

$E$ is open iff $E = E^0$.

(Defn) Limit Point :- Let $E \subset X$. A point $p \in X$ is called a limit point of $E$ if every n'hood of $p$ contains a point of $E$ other than $p$.

The set of all limit points of $E$ is called the derived set of $E$ and is denoted by $E'$.

A subset $E$ of $X$ is said to be closed iff every limit point of E lies in E, i.e. if $E' \subset E$.

The Closure of a set E, denoted by $\bar{E}$, is set of all Points of $\bar{E}$ and its limit points

$\bar{E}$ is always closed and E is closed iff $E = \bar{E}$.

(Theorem) (a) Arbitrary union of open sets is open
(b) Finite intersection of open sets is open
(c) Arbitrary intersection of closed sets is closed
(d) Finite union of closed sets is closed.

Proof: (a) Let $\{G_\alpha\}$ be a family of open sets and let $G = \cup G_\alpha$.

Let $p \in G$. Then $p \in G_\alpha$ for some $\alpha$. Since $G_\alpha$ is open there exists a n'hood $N_r(p)$ of p such that $N_r(p) \subset G_\alpha$.

Then $N_r(p) \subset G$. Hence G is open.

(b) Let $G_1, G_2, -----, G_n$ be open sets and let $G = G_1 \cap ------ \cap G_n$.

Let $p \in G$. Then $p \in G_i$ for $i = 1, 2, ---- n$.

This means that there exist real number $r_1, r_2, ------ r_n > 0$

$N_{r_i}(p) \subset G_i$

If $r = \min\{r_1, \ldots, r_n\}$. Then
$N_r(p) \subset G_i$ for $i = 1, 2, \ldots n$.
So, $N_r(p) \subset G$.
Hence $G$ is open.

(Defn) Sequence: A sequence is a function defined on rational number.
Let $\{p_n\}$ be a sequence is a metric space $X$. Then $\{p_n\}$ is said to converge in $X$ if there exists a point $p \in X$ satisfying the condition that for every $\varepsilon > 0$ there exists a rational number $N$ s.t.
$n \geq N \Rightarrow d(p_n, p) < \varepsilon$.

A sequence $\{p_n\}$ is a metric space $X$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exists rational number $N$ s.t.
$m \geq N, n \geq N \Rightarrow d(p_m, p_n) < \varepsilon$.

(Defn) Continuity:- Let $f : X \to Y$ be a ~~map~~ mapping of a metric space $X$ into a metric space $Y$. Let $x \in X$. Then $f$ is said to be continuous at $x$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that,
$d(f(x), f(y)) < \varepsilon$. When every $x, y \in X$ and $d(x, y) < \delta$.

If $f$ is Continuous at every Point $x \in X$ then it is said to be Continuous on $X$.

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(Theorem) Let $f: X \to Y$. Then $f$ is Continuous on $X$ iff $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.

'93 Or, Q.No> Give the characterization of Continuity of function from one metric space to another in terms of open sets.

Proof:- Let $f: X \to Y$ be a Continuous mapping of a metric space $X$ into a metric space $Y$. ~~Suppo~~ Suppose that $V$ is an open subset of $Y$. We consider $f^{-1}(V)$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since $V$ is open, there is $\varepsilon > 0$ such that $N_\varepsilon(f(x)) \subset V$. Again the continuity of $f$ at $x$ implies that there exists $\delta > 0$ such that $f(N_\delta(x)) \subset V$. This means that $N_\delta(x) \subset f^{-1}(V)$. So, $x$ is an interior point of $f^{-1}(V)$. This proves that $f^{-1}(V)$ is open in $X$.

Conversely, suppose that $f^{-1}(V)$ is open in $X$ whenever $V$ is open in $Y$. Let $x \in X$ and $\varepsilon > 0$ be given. Then $f(x) \in Y$. We consider $N_\varepsilon(f(x))$. This is an open subset of $Y$. So according to one assumption $f^{-1}(N_\varepsilon(f(x)))$ is open in $X$. Also $x \in f^{-1}(N_\varepsilon(f(x)))$. This means that there exists $\delta > 0$ such that $N_\delta(x) \subset f^{-1}(N_\varepsilon(f(x)))$ i.e., $f(N_\delta(x)) \subset N_\varepsilon(f(x))$.

In other words, $y \in X$, $d(y, x) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

Hence $f$ is continuous at $x$. Since $x$ is arbitrary $f$ is open on $X$.

(Def^n) Uniform Continuity:- Let $f: X \to Y$ be a mapping of a metric space $X$ into a metric space $Y$. Then $f$ is said to be uniformly continuous on $X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $x, y \in X$ and $d(x, y) < \delta$.

QNo> Let $f$ be a continuous mapping of a compact metric space $X$ into a metric space $Y$. Then $f$ is uniformly continuous on $X$.

Proof:- Suppose that $f$ is a mapping of a compact metric space $X$ into a metric space $Y$. Let $\varepsilon > 0$ be given. Since $f$ is continuous at every point $x \in X$, with each $x \in X$ we can associate a real number $\delta(x) > 0$ such that,

$$y \in X, d(x, y) < \delta(x) \Rightarrow d(f(x), f(y)) < \frac{\varepsilon}{2} \quad \text{--- (1)}$$

Let $J(x) = \{y \in X : d(y, x) < \frac{1}{2}\delta(x)\}$. Then the collection $\{J(x)\}_{x \in X}$ is an open cover of $X$. Since $X$ is compact, this open cover has a finite subcover. This means that there exist finite no of Points $x_1, x_2, ----x_n$ such that, $X \subset J(x_1) \cup ------ \cup J(x_n)$.

We Put $\delta = \frac{1}{2} \min\{\delta(x_1), \delta(x_2), -------, \delta(x_n)\}$.
Then $\delta > 0$.

Suppose that $x, y \in X$ and $d(x,y) < \delta$. Then there exists $m$ with $1 \leq m \leq n$ such that $x \in J(x_m)$.
This means that $d(x_1, x_m) < \frac{1}{2}\delta(x_m)$.
We also have

$$d(y, x_m) \leq d(y_1, x) + d(x_1, x_m) < \delta + \frac{1}{2}\delta(x_m) < \delta(x_m)$$

So, $d(f(y), f(x) \leq d(f(y), f(x_m)) + d(f(x_m), f(x))$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f$ is uniformly Continuous on $X$.

Example of a Continuous mapping which is not uniformly Continuous.

Let $E$ be a bounded non-Compact Subset of $R$
$E = (0, 1]$

We define $f(x) = \frac{1}{x - 0}$

Then $f$ is Continuous but not uniformly Continuous.